# Best - Fit Polynomials

( Section 5.6 )

Recall: If we have $n$ points in $\mathbb{R}^2$ with no repeated $x$ values, then there is a polynomial of degree $n-1$ that passes through all $n$ points, called an interpolating polynomial.

What if we want a polynomial of lower degree that only gets "close" to these points, like a line?

**Example 1:** (best-fit line, 3 points)

Given the points $(1,3)$, $(2,-1)$, and $(6,5)$, try to find a line through these three points.

Slope between the first two points:

$$\frac{3-(-1)}{1-2} = -4$$

Slope between the second two points:

$$\frac{-1-5}{2-6} = \frac{6}{4} \neq -4$$

So no such line exists!

Pretend there was such a line $y = mx + b$

with

$$3 = m + b$$
$$-1 = 2m + b$$
$$5 = 6m + b$$

Rewrite as a matrix-vector equation:

$$\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 6 \end{bmatrix}}_{\text{is this invertible?}} \begin{bmatrix} b \\ m \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 6 \end{bmatrix}$$ is not a square matrix,

so not invertible.

$$\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix}$$

Multiply both sides by

$$A^t = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 6 \end{bmatrix}$$ on the left :

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 31 \end{bmatrix}$$

$$\underset{2\times 3}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 6 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 6 \end{bmatrix}} = \begin{bmatrix} 3 & 9 \\ 9 & 41 \end{bmatrix}$$

gives a $2\times 2$

We get the new equation

$$\begin{bmatrix} 7 \\ 31 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 9 \\ 9 & 41 \end{bmatrix}}_{\text{is this invertible?}} \begin{bmatrix} b \\ m \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 3 & 9 \\ 9 & 41 \end{bmatrix}\right) = 123 - 81 = 42 \neq 0$$

So $\begin{bmatrix} 3 & 9 \\ 9 & 41 \end{bmatrix}$ is invertible.

Multiply both sides by the inverse

$$\frac{1}{42}\begin{bmatrix} 41 & -9 \\ -9 & 3 \end{bmatrix}$$

Formally, this is

$$(A^tA)^{-1}\begin{bmatrix}7\\31\end{bmatrix} = \cancel{(A^tA)^{-1}(A^tA)}\begin{bmatrix}b\\m\end{bmatrix}$$

identity matrix

$$\begin{bmatrix}b\\m\end{bmatrix} = (A^tA)^{-1}\begin{bmatrix}7\\31\end{bmatrix}$$

$$\begin{bmatrix}b\\m\end{bmatrix} = \frac{1}{42}\begin{bmatrix}41 & -9\\-9 & 3\end{bmatrix}\begin{bmatrix}7\\31\end{bmatrix}$$

$$\begin{bmatrix}b\\m\end{bmatrix} = \frac{1}{21}\begin{bmatrix}4\\15\end{bmatrix}$$

so we get

$$\boxed{y = \frac{15}{21}x + \frac{4}{21}}.$$

Does this line pass through any of our original points?

NO.

What it does do is minimize the average distance from the line to the points.

**Procedure:** To get the "best-fit" polynomial to a collection of points $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ in $\mathbb{R}^2$, start with a polynomial of degree less than or equal to $n-1$, call it

$$p(x) = \sum_{i=0}^{m} c_i x^i$$

where $m$ is the degree and $c_0, c_1, c_2, \cdots c_m$ are real numbers.

Step 1 Plug your points in:

$$y_1 = p(x_1) = \sum_{i=0}^{m} c_i x_1^i$$

$$y_2 = p(x_2) = \sum_{i=0}^{m} c_i x_2^i$$

$$\vdots$$

$$y_n = p(x_n) = \sum_{i=0}^{m} c_i x_n^i .$$

Step 2 Make a matrix-vector equation

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = A \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{bmatrix}$$

where $A$ is the matrix

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^m \\ 1 & x_2 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^m \end{bmatrix}$$

$A$ is $n \times (m+1)$

Not invertible unless $m = n-1$ !

Step 3 Multiply both sides by

$A^t$ to get

$$A^t \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = (A^tA) \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{bmatrix}$$

Step 4 If $A^tA$ is invertible, multiply both sides of the equality by $(A^tA)^{-1}$:

$$(A^tA)^{-1} A^t \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{bmatrix}$$

**Step 5:** Write down your polynomial.

Wolfram Alpha

Check using "best fit line (points)"

or "best fit quadratic (points)"

etc., depending on the degree of your polynomial.

**Example 2:** (more points) Find the best-fit line through the points $(1,5)$, $(-2,7)$, $(9,6)$, and $(0,4)$.

**Solution:** Write the line as $y=p(x)=mx+b$.

**Step 1:** Plug in points.

$$5 = p(1) = m+b$$
$$7 = p(-2) = -2m+b$$
$$6 = p(9) = 9m+b$$
$$4 = p(0) = b$$

Step 2: Matrix - Vector equation

$$\begin{bmatrix} 5 \\ 7 \\ 6 \\ 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 9 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} b \\ m \end{bmatrix}$$

Step 3: Multiply both sides by $A^t$

$$A^t = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 9 & 0 \end{bmatrix}$$

$$A^t \begin{bmatrix} 5 \\ 7 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 22 \\ 45 \end{bmatrix}$$

$$A^t A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 9 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 9 \\ 1 & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 4 & 8 \\ 8 & 86 \end{bmatrix}}_{\text{invertible!}}$$

We get

$$\begin{bmatrix} 22 \\ 45 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 86 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix}$$

**Step 4:** Multiply both sides by $(A^tA)^{-1}$

$$(A^tA)^{-1}\begin{bmatrix} 22 \\ 45 \end{bmatrix} = \begin{bmatrix} b \\ m \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 \\ 8 & 86 \end{bmatrix}^{-1}\begin{bmatrix} 22 \\ 45 \end{bmatrix} = \begin{bmatrix} b \\ m \end{bmatrix}$$

$$\begin{bmatrix} b \\ m \end{bmatrix} = \frac{1}{70}\begin{bmatrix} 383 \\ 1 \end{bmatrix}$$

**Step 5:** The best fit line is

$$\boxed{y = \frac{1}{70}x + \frac{383}{70}}$$

**Example 3:** (best-fit quadratic) Find the best-fit quadratic through the points $(1,5)$, $(-2,7)$, $(9,6)$, and $(0,4)$.

**Solution:** Write our polynomial as

$$y = p(x) = ax^2 + bx + c.$$

**Step 1:** Plug in the points

$$5 = p(1) = a + b + c$$

$$7 = p(-2) = 4a - 2b + c$$

$$6 = p(9) = 81a + 9b + c$$

$$4 = p(0) = c$$

**Step 2:** Matrix-vector equation

$$\begin{bmatrix} 5 \\ 7 \\ 6 \\ 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 4 \\ 1 & 9 & 81 \\ 1 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} c \\ b \\ a \end{bmatrix}$$

**Step 3:** Multiply both sides by $A^t$.

$$A^t = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 9 & 0 \\ 1 & 4 & 81 & 0 \end{bmatrix}$$

$$A^t \begin{bmatrix} 5 \\ 7 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 22 \\ 45 \\ 519 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 9 & 0 \\ 1 & 4 & 81 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 4 \\ 1 & 9 & 81 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^tA = \underbrace{\begin{bmatrix} 4 & 8 & 86 \\ 8 & 86 & 722 \\ 86 & 722 & 6578 \end{bmatrix}}_{\text{invertible!}}$$

**Step 4:** Multiply both sides by $(A^tA)^{-1}$:

$$(A^tA)^{-1}\begin{bmatrix} 22 \\ 45 \\ 519 \end{bmatrix} = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$$

We get

$$\begin{bmatrix} c \\ b \\ a \end{bmatrix} = \frac{1}{9510} \begin{bmatrix} 46884 \\ -6857 \\ 890 \end{bmatrix}$$

Step 5: The polynomial is

$$\boxed{y = \frac{890}{9510} x^2 - \frac{6857}{9510} + \frac{46{,}884}{9510}}$$

Q: What's going on here? What are we actually doing with these operations?

Warning: Use $A^tA$, not $AA^t$!